

# Thermal conductivities of one-dimensional anharmonic/nonlinear lattices: renormalized phonons and effective phonon theory

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Heat transport in low-dimensional systems has attracted enormous attention from both theoretical and experimental aspects due to its significance to the perception of fundamental energy transport theory and its potential applications in the emerging field of phononics: manipulating heat flow with electronic analogs. We consider the heat conduction of one-dimensional nonlinear lattice models. The energy carriers responsible for the heat transport have been identified as the renormalized phonons. Within the framework of renormalized phonons, a phenomenological theory, *effective phonon theory*, has been developed to explain the heat transport in general one-dimensional nonlinear lattices. With the help of numerical simulations, it has been verified that this effective phonon theory is able to predict the scaling exponents of temperature-dependent thermal conductivities *quantitatively and consistently*.

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## I. INTRODUCTION

The Fourier's heat conduction law states that the heat current flowing through a system is proportional to the temperature gradient imposed on the two ends of the system:  $j = -\kappa \nabla T$ , where  $j$  is the heat current,  $\nabla T$  is the temperature gradient and the proportionality  $\kappa$  is defined as the thermal conductivity. A diffusive or normal heat conduction requires that  $\kappa$  is independent to the system length  $L$  in the thermodynamical limit  $L \rightarrow \infty$ . However, the discovery of anomalous heat conduction [1] that  $\kappa \propto L^\alpha$  with  $0 < \alpha < 1$  for one-dimensional Fermi-Pasta-Ulam  $\beta$  (FPU- $\beta$ ) lattice has casted doubt on the validity of Fourier's heat conduction law on low-dimensional systems and stimulated intensive studies on this issue from both theoretical [2–6] and experimental [7] attempts. On the other hand, it has been demonstrated that the nonlinearity can be utilized to design novel nanoscale solid-state thermal devices such as thermal diodes[8–11], thermal transistors[12], thermal logic gates[13] and thermal memories[14, 15] which gives birth to the innovating field of phononics: manipulating/controlling heat flow and processing information with phonons[16]. The distinctive and unique transport property of low-dimensional system has posted great challenge to the complete microscopic transport theory. Therefore, any theoretical attempt towards a thorough understanding of the heat transport in general one-dimensional nonlinear lattice systems is timely and highly desirable.

In order to reveal the physical mechanism underlying the heat transport in one-dimensional nonlinear lattices, the energy carriers responsible for heat transport must be identified on the first place. Although the linear Harmonic lattice can only sustain the vibrations of phonons, it has been known that there are more than one type of excitation modes in nonlinear lattices, i.e. the renormalized phonons[17–24], solitons[25–28] and breathers[29, 30]. In particular, the stable solitons which almost do not interact with each other has been argued to be the origin of the anomalous heat conduction found in FPU- $\beta$  lattice[31–33]. The numerical calculations for the sound velocities of energy carriers have also been found to follow the prediction of soliton velocities[32]. However, these calculations cannot exclude the possibility of renormalized phonons as energy carriers since the prediction of sound velocities of renormalized phonons is not far from the prediction of solitons[22]. It will be necessary to determine which excitation mode is the energy carrier for nonlinear lattices and more accurate calculations need to be performed to uncover this confusion[34].

For the past decades, enormous efforts have been focused on the study of normal or anomalous heat conduction, i.e. size-dependent thermal conductivities of one-dimensional nonlinear lattices[1–6]. On the contrary, there are only a few works dealing with the temperature-dependent behaviors of heat conduction [23, 24, 32, 35–41] which should

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be more relevant to experimental investigations. The temperature dependence of heat conduction is caused by the nonlinearity as the linear Harmonic lattice cannot display temperature modulated response. Furthermore, the novel thermal devices such as the thermal diodes are rooted on the fact of temperature or nonlinearity modulated phonon spectrum. The understanding of the temperature-dependent behavior of heat conduction for nonlinear lattices is thus of both theoretical and experimental importance. A transport theory based on the correct energy carriers will definitely help us to unravel the underlying physical mechanism for the heat transport in low-dimensional nonlinear lattices.

The paper is organized as the follows: in Sec. II the general one-dimensional nonlinear lattice models will be introduced and the concept of renormalized phonons will be discussed. The sound velocities of energy carriers will be analyzed and determined through detailed numerical calculations. Sec. III will then present the phenomenological theory of effective phonon theory in the framework of renormalized phonons. The predictions for temperature-dependent thermal conductivities from effective phonon theory will be compared with Non-Equilibrium Molecular Dynamics (NEMD) simulation results. We will give conclusions and summaries in Sec. IV.

## II. NONLINEAR LATTICES AND RENORMALIZED PHONONS

The one-dimensional nonlinear lattice Hamiltonian has a general form:

$$H = \sum_{i=1}^N \left[ \frac{p_i^2}{2} + V(x_{i+1}, x_i) + U(x_i) \right], \quad (1)$$

where  $p_i$  and  $x_i$  denote the momentum and deviation from equilibrium position for the  $i$ -th atom, respectively. For simplicity, the periodic boundary condition  $x_1 \equiv x_{N+1}$  is often applied where  $N$  is the number of atoms or the size of lattice in dimensionless unit since  $L = Na$  where  $a$  is the lattice constant and can be set as unity[16]. The inter-atom potential  $V(x_{i+1}, x_i)$  and the on-site potential  $U(x_i)$  also assume the following general expressions:

$$V(x_{i+1}, x_i) = \sum_{s=2}^{\infty} v_s \frac{(x_{i+1} - x_i)^s}{s}, \quad U(x_i) = \sum_{s=2}^{\infty} u_s \frac{x_i^s}{s} \quad (2)$$

For simplicity, the coefficients  $v_s$  and  $u_s$  only take two values 0 and 1. The celebrated FPU- $\beta$  lattice

$$H = \sum_{i=1}^N \left[ \frac{p_i^2}{2} + \frac{(x_{i+1} - x_i)^2}{2} + \frac{(x_{i+1} - x_i)^4}{4} \right] \quad (3)$$

has two nonvanishing coefficients  $v_2 = v_4 = 1$  while the most studied on-site  $\phi^4$  lattice

$$H = \sum_{i=1}^N \left[ \frac{p_i^2}{2} + \frac{(x_{i+1} - x_i)^2}{2} + \frac{x_i^4}{4} \right] \quad (4)$$

has two nonvanishing coefficients with  $v_2 = u_4 = 1$ .

For the Harmonic lattice, it is well-known that the Hamiltonian can be decomposed into the energy sum of normal modes (phonons) under the transformation  $q_k = \sum_{i=1}^N S_{ki} x_i$  and  $p_k = \sum_{i=1}^N S_{ki} p_i$  where the matrix element  $S_{ki} = \frac{1}{\sqrt{N}} (\sin 2\pi k i / N + \cos 2\pi k i / N)$ . The Hamiltonian in position space can thus be transformed into normal mode space as

$$H = \sum_{i=1}^N \left[ \frac{p_i^2}{2} + \frac{(x_{i+1} - x_i)^2}{2} \right] \rightarrow H = \sum_{k=1}^N \left[ \frac{p_k^2}{2} + \frac{\omega_k^2 q_k^2}{2} \right], \quad \omega_k = 2 \sin \frac{\pi k}{N} \quad (5)$$

where  $\omega_k$  is the phonon frequency. The resulted total Hamiltonian in phonon space is the energy summation of every independent phonon modes. Therefore, the Harmonic lattice has no phonon-phonon interaction and is called linear lattice in this sense.

As for the nonlinear lattice, there does not exist any transformation which can transfer the lattice Hamiltonian into the sum of independent phonon modes. However, the nonlinear lattice Hamiltonian can be expressed *approximately* as the energy summation of phonon modes with renormalized frequencies due to the nonlinearity.

The connection between the renormalized phonon frequency and the nonlinearity originates from the generalized equipartition theorem:

$$k_B T = \left\langle q_k \frac{\partial H}{\partial q_k} \right\rangle \quad (6)$$

where  $k_B$  is the Boltzmann constant and  $\langle \cdot \rangle$  denotes ensemble average. For Harmonic lattice, it is easy to see from Eq. (5) that

$$k_B T = \omega_k^2 \langle q_k^2 \rangle \quad (7)$$

which is the familiar result that every degree of freedom (phonon mode) shares the same amount of energy. For nonlinear lattice, a similar relation holds for the renormalized phonon modes under the mean-field approximation[22]:

$$k_B T \approx \hat{\omega}_k^2 \langle q_k^2 \rangle \quad (8)$$

where the renormalized phonon frequencies can be defined as

$$\begin{aligned} \hat{\omega}_k &= \sqrt{\alpha \cdot \omega_k^2 + \gamma} \\ \alpha &= \sum_{s=2}^{\infty} v_s \frac{\left\langle \sum_{i=1}^N (x_{i+1} - x_i)^s \right\rangle}{\left\langle \sum_{i=1}^N (x_{i+1} - x_i)^2 \right\rangle} \\ \gamma &= \sum_{s=2}^{\infty} u_s \frac{\left\langle \sum_{i=1}^N x_i^s \right\rangle}{\left\langle \sum_{i=1}^N x_i^2 \right\rangle} \end{aligned} \quad (9)$$

and the renormalization coefficients  $\alpha$  and  $\gamma$  encode the information of inter-atom potential and on-site potential, respectively. For the simple Harmonic lattice with  $v_2 = 1$ , the renormalization coefficient  $\alpha = 1$  and  $\gamma = 0$ , the renormalized phonon frequencies  $\hat{\omega}_k = \omega_k$  thus recovers the original phonon dispersion relation.

The nonlinear lattices can be classified into two categories: with or without on-site potential. For lattices without on-site potential as  $U(x_i) = 0$ , we have  $\gamma = 0$  from Eq. (9). Therefore, the branch of renormalized phonon modes is acoustic-like with

$$\hat{\omega}_k = \sqrt{\alpha} \cdot \omega_k \quad (10)$$

For lattices with on-site potential as  $U(x_i) \neq 0$ , the renormalization coefficient  $\gamma \neq 0$  and the resulted phonon branch is optic-like.

In order to understand the physical mechanism of the heat transport, we need first to determine the energy carriers in nonlinear lattices. Besides the renormalized phonons, other excitation modes such as solitons and breathers existing in nonlinear lattices are also possible candidates for energy carriers. One of the best way to identify the energy carriers is to calculate the temperature-dependent energy transport speeds and compare them with the theoretical predictions for the sound velocities of different excitation modes.

The sound velocity of renormalized phonons can be derived from  $c_s = \frac{\partial \hat{\omega}_q}{\partial q}|_{q=0}$  where we have applied the substitution  $q = 2\pi k/N$  and  $\omega_q = 2 \sin q/2$  in the continuous limit. For the well-known FPU- $\beta$  lattice with  $v_2 = v_4 = 1$ , the sound velocity of renormalized phonons can be obtained from Eq. (5), (9) and (10):

$$c_s = \sqrt{\alpha} = \sqrt{1 + \frac{\int_0^\infty dx \cdot x^4 e^{-(x^2/2+x^4/4)/T}}{\int_0^\infty dx \cdot x^2 e^{-(x^2/2+x^4/4)/T}}} \quad (11)$$

where dimensionless unit has been used and the Boltzmann constant  $k_B$  has been set as unity. The sound velocity depends on the temperature, or equivalently the strength of nonlinearity. In the low temperature limit  $T \rightarrow 0$ ,  $c_s \rightarrow 1$  as expected for Harmonic lattice. In the high temperature region  $T \gg 1$ , the sound velocity  $c_s \approx 1.22T^{1/4}$  exhibits strong nonlinear dependence. On the other hand, the soliton theory predicts a temperature dependent sound velocity as

$$c_s^3 \sqrt{c_s^2 - 1} = \eta T \quad (12)$$

where  $\eta$  is a fitting parameter. By tuning  $\eta = 2.215$ , the sound velocity from soliton theory of Eq. (12) coincides with the prediction from renormalized phonons of Eq. (11) in low and high temperature region. However, these

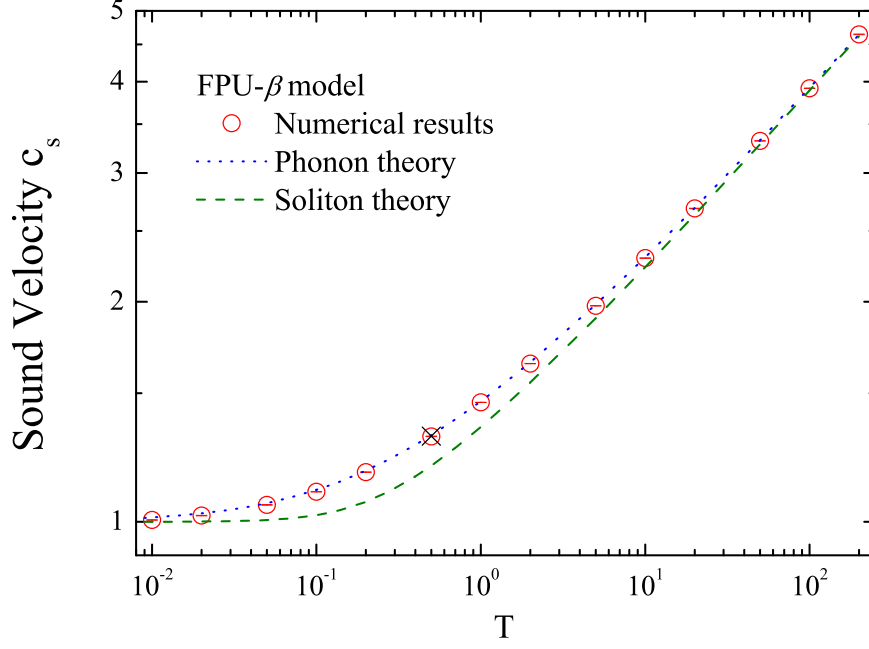


FIG. 1: (color online). Sound velocity  $c_s$  as the function of temperature  $T$  for FPU- $\beta$  lattice. The circles are the numerical data. The dotted line is the theoretical prediction from renormalized phonons from Eq. (11). The dashed line is the theoretical prediction for solitons derived from the formula of Eq. (12) with  $\eta = 2.215$ . The cross symbol is the numerical data for  $T = 0.5$  obtained in Ref. [42]. Adapted from Ref. [34].

two predictions deviate from each other in the intermediate temperature region and can only be verified by accurate numerical calculations[42].

In Fig. 1, the numerically calculated sound velocity  $c_s$  for FPU- $\beta$  lattice has been compared with the prediction from the theory of renormalized phonons and solitons[34], respectively. It can be clearly seen that the soliton prediction deviates from numerical data in the intermediate temperature region while the prediction of renormalized phonons matches perfectly with the numerical data in all temperature regions. This is a clear evidence that the energy carriers in the FPU- $\beta$  lattice are the renormalized phonons rather than the previously believed solitons.

To demonstrate the validity and consistency of the renormalized phonon formulation, the other three  $H_n$  models with  $n = 3, 4, 5$  have been considered with the following Hamiltonian:

$$H_n = \sum_{i=1}^N \left[ \frac{p_i^2}{2} + \frac{|x_{i+1} - x_i|^n}{n} \right] \quad (13)$$

The  $H_4$  model is just the high temperature limit of FPU- $\beta$  lattice. In the framework of renormalized phonon formulation, the sound velocities of  $H_n$  model can be expressed in a compact form:

$$c_s = \sqrt{\alpha} = \sqrt{\frac{\int_0^\infty dx \cdot x^n e^{-\frac{x^n}{nT}}}{\int_0^\infty dx \cdot x^2 e^{-\frac{x^n}{nT}}}} = \sqrt{\frac{\Gamma(\frac{n+1}{n})}{\Gamma(\frac{3}{n})}} (nT)^{\frac{1}{2} - \frac{1}{n}} \quad (14)$$

In Fig. 2, the numerically calculated sound velocities are compared with the predictions for renormalized phonons for  $H_n$  lattices with  $n = 3, 4$ , and 5. Again, the *quantitative* agreements have been found for all three models which we considered.

The above examples are for the lattices without on-site potential. For the lattices with on-site potential such as  $\phi^4$  lattice, there are numerical results suggesting that the energy carriers are also renormalized phonons[43].

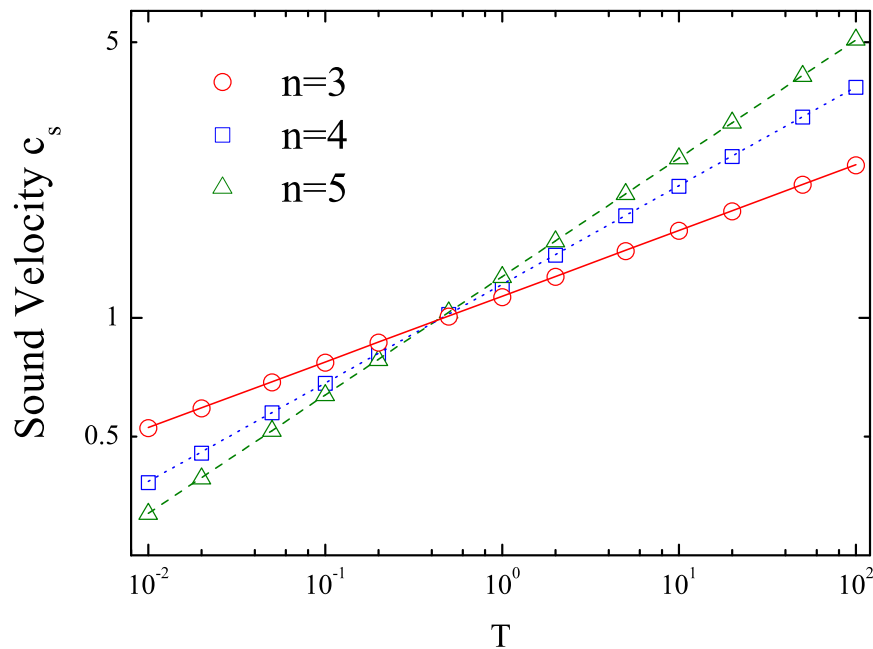


FIG. 2: (color online). Sound velocity  $c_s$  as the function of temperature  $T$  for  $H_n$  lattices with  $n = 3, 4$ , and  $5$ . The symbols are the numerical data, whereas the lines are the prediction for renormalized phonons from Eq. (14). Adapted from Ref. [34].

### III. EFFECTIVE PHONON THEORY

So far as we know, there is no microscopic transport theory which is able to explain the heat transport in nonlinear lattices with strong nonlinearity. The perturbation theory will become invalid as the nonlinear term cannot be ignored anymore in this case. For example, no existing theory can even predict the temperature dependence of thermal conductivities for general nonlinear lattices. A phenomenological theory possessing such kind of prediction power is thus highly desirable.

The thermal conductivity for solids is usually described by the Deybe formula:

$$\kappa = \frac{c}{2\pi} \int_0^{2\pi} dq v_q^2 \tau_q \quad (15)$$

where  $c$  denotes the specific heat,  $v_q$  denotes the group velocity and  $\tau_q$  denotes the relaxation time for phonon mode  $q$ . The key issue for the Deybe formula is then to determine the relaxation time  $\tau_q$ .

For nonlinear lattices, we have shown the evidences that the energy carriers should be the renormalized phonons. Therefore, the transport theory should be constructed with the basis of renormalized phonons. In consideration of the nonlinearity and anomalous size-dependence of thermal conductivity, we propose the modified Deybe formula in the framework of renormalized phonons:

$$\kappa = \frac{c}{2\pi} \int_0^{2\pi} dq v_q^2 \tau_q P(q) \quad (16)$$

where  $P(q)$  is a weight factor with normalization condition  $\frac{1}{2\pi} \int_0^{2\pi} dq P(q) = 1$ ,  $v_q$  is the group velocity of renormalized phonons as  $v_q = \partial \hat{\omega}_q / \partial q$ , and  $\tau_q$  is the relaxation time for renormalized phonons which assumes the following proportionality:

$$\tau_q \propto \frac{1}{\epsilon} \cdot \frac{2\pi}{\hat{\omega}_q} \quad (17)$$

where  $\hat{\omega}_q$  is the frequency for renormalized phonons and the dimensionless parameter  $\epsilon$  represents the strength of

nonlinearity defined as:

$$\epsilon = \frac{|\langle E_n \rangle|}{\langle E_l + E_n \rangle}, \quad 0 \leq \epsilon \leq 1 \quad (18)$$

where  $E_l$  denotes the linear potential energy and  $E_n$  denotes the nonlinear potential energy. The parameter  $\epsilon$  describes the ratio of nonlinear potential energy to the total potential energy and is independent of mode  $q$ .

The introduction of the weight factor  $P(q)$  is to be consistent with the anomalous size-dependent thermal conductivities for lattices without on-site potential such as the FPU- $\beta$  lattice. If we assume  $P(q) \propto 1/q^\delta$  in the long wave-length limit  $q \rightarrow 0$ , it is straight forward to obtain the size-dependent  $\kappa$  as

$$\kappa \propto \int_{0^+}^{2\pi} dq v_q^2 \tau_q P(q) \propto \int_{0^+}^{2\pi} dq \frac{1}{\omega_q} \cdot \frac{1}{q^\delta} \propto q^{-\delta} \propto N^\delta \quad (19)$$

the lower integration limit  $0^+$  describes the fact that the lattice size is always finite with length  $N$  and  $\omega_q \propto q \propto 1/N$  in the long wave-length limit  $q \rightarrow 0$  due to the finite size effect. The exact value of exponent  $\delta$  is still unresolved and is argued to be  $2/5$  from mode-coupling theory or  $1/3$  from hydrodynamic theory[3]. However, there are one exception that the momentum conserved one-dimensional coupled rotator model without on-site potential does exhibit normal heat conduction behavior[44, 45]. Most recently, some numerical works even suggest that the momentum conserved one-dimensional lattice models with asymmetric interactions might also follow the Fourier's normal heat conduction law[46, 47].

It should be noticed that the introduction of  $P(q)$  simultaneously explains the normal heat conduction for lattices with on-site potential. The non-zero renormalization coefficient  $\gamma$  shifts the renormalized phonon frequency from zero to  $\hat{\omega}_q = \sqrt{\gamma}$  in the long wave-length limit  $q \rightarrow 0$ . The integral in last equation becomes

$$\kappa \propto \int_{0^+}^{2\pi} dq \frac{1}{\hat{\omega}_q} \cdot \frac{1}{q^\delta} < \frac{1}{\sqrt{\gamma}} \int_{0^+}^{2\pi} dq \frac{1}{q^\delta} \quad (20)$$

One should notice that  $\delta < 1$  due to the constriction of normalization condition for  $P(q)$  as  $\frac{1}{2\pi} \int_0^{2\pi} dq P(q) = 1$ . The integral is non-divergent and the thermal conductivities for lattices with on-site potential is finite and obeys Fourier's heat conduction law which is consistent with previous results[31, 35, 48].

In this paper, we will only focus on the temperature dependent thermal conductivities for nonlinear lattices without on-site potential. The renormalization coefficient  $\gamma = 0$  and the renormalization phonon frequency  $\hat{\omega}_q = \sqrt{\alpha}\omega_q$  where temperature dependent part  $\alpha$  and mode dependent part  $\omega_q$  are separate. From Eq. (16) and (17), the temperature dependence of thermal conductivity can be fully described by the simple formulation

$$\kappa(T) \propto \frac{\sqrt{\alpha}}{\epsilon} \propto \frac{c_s}{\epsilon} \quad (21)$$

where the renormalization coefficient  $\alpha$  (or the sound velocity  $c_s$ ) and nonlinearity strength  $\epsilon$  are fully determined by the lattice Hamiltonian. One should also notice that these two parameters only depend on temperature and are mode independent. In another word, the size- and temperature-dependences of the thermal conductivities are separate, which enables us to discuss the temperature-dependence of the thermal conductivities alone without worrying about the annoying size-dependence. This effect has been verified by our numerical simulations (not shown here).

For the FPU- $\beta$  lattice with Hamiltonian of Eq. (3), the temperature dependence of  $c_s$  and  $\epsilon$  exhibits scaling relations at both low and high temperature limits. As we have discussed above, the sound velocity  $c_s$  shows the following temperature dependence:

$$c_s \propto \text{const}, T \ll 1; \quad c_s \propto T^{1/4}, T \gg 1 \quad (22)$$

From definition of Eq. (18), the nonlinearity strength  $\epsilon$  for FPU- $\beta$  lattice can be expressed as

$$\epsilon = \frac{\langle \sum_i (x_{i+1} - x_i)^4 / 4 \rangle}{\langle \sum_i (x_{i+1} - x_i)^2 / 2 + \sum_i (x_{i+1} - x_i)^4 / 4 \rangle} \quad (23)$$

At low temperature limit, the quartic potential term is a small term comparing to the quadratic potential term. The whole system can be described by a harmonic lattice plus a small perturbation. From equipartition theorem for harmonic lattice, we thus have  $\langle \sum_i (x_{i+1} - x_i)^2 \rangle \approx N k_B T$  for the FPU- $\beta$  lattice at low temperature limit. This also

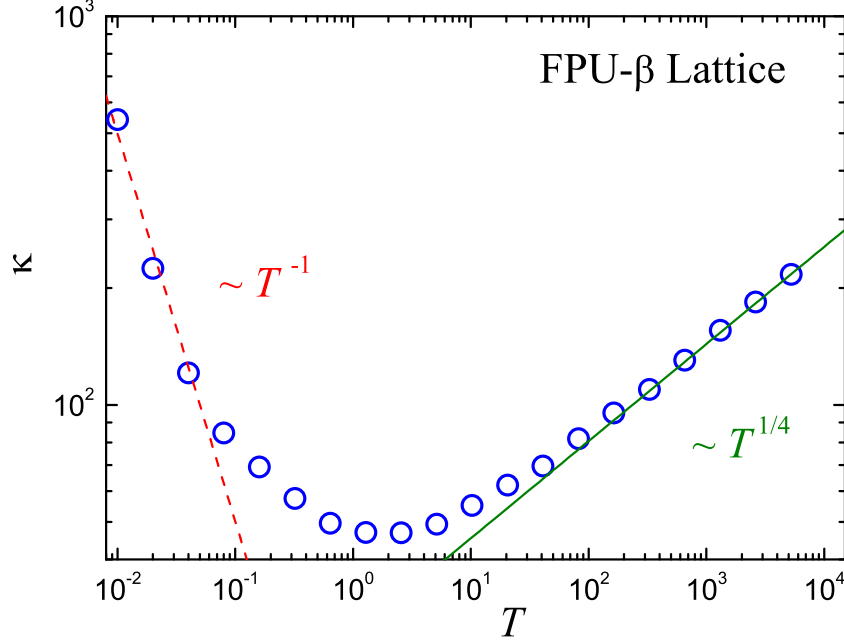


FIG. 3: (color online). Thermal conductivity  $\kappa$  as the function of temperature  $T$  for FPU- $\beta$  lattice. The lattice size is set as  $N = 1000$ . The circles are the numerical data from NEMD simulations.

gives rise to the scaling relation for the quartic potential term, i.e.  $\langle \sum_i (x_{i+1} - x_i)^4 \rangle \propto T^2$ . Thus the nonlinearity strength  $\epsilon$  follows

$$\epsilon \approx \frac{\langle \sum_i (x_{i+1} - x_i)^4 / 4 \rangle}{\langle \sum_i (x_{i+1} - x_i)^2 / 2 \rangle} \propto T \quad (24)$$

On the other hand, the quadratic potential term is a small term at high temperature limit and the nonlinearity strength  $\epsilon$  approaches to the upper limit 1. The entire temperature dependence of  $\epsilon$  can be summarized as:

$$\epsilon \propto T, T \ll 1; \quad \epsilon \propto \text{const}, T \gg 1 \quad (25)$$

From Eq. (21), (22) and (25), we can derive the temperature dependence of  $\kappa$  for FPU- $\beta$  lattice at both low and high temperature:

$$\kappa(T) \propto \frac{1}{T}, T \ll 1; \quad \kappa(T) \propto T^{1/4}, T \gg 1 \quad (26)$$

In Fig. 3, the numerical results of thermal conductivities  $\kappa$  calculated by Non-Equilibrium Molecular Dynamics (NEMD) simulations are plotted for FPU- $\beta$  lattices. It can be seen that the scaling of the thermal conductivities follows the relations  $\kappa \propto 1/T$  at low temperature limit and  $\kappa \propto T^{1/4}$  at high temperature limit, respectively. The temperature dependence of  $\kappa$  for FPU- $\beta$  lattice can thus be *consistently* explained by our effective phonon theory at both low and high temperature limits.

To further verify the predictions from effective phonon theory, we also consider the  $H_n$  lattices with Hamiltonian of Eq. (13). The fact that the potential energy of these  $H_n$  lattices is fully nonlinear gives rise to the special property for the nonlinearity strength as  $\epsilon = 1$ . From Eq. (14), the temperature dependence of the sound velocity  $c_s$  follows  $c_s \propto T^{1/2-1/n}$ . The thermal conductivities  $\kappa$  for  $H_n$  lattices can be derived as:

$$\kappa \propto c_s \propto T^{\frac{1}{2}-\frac{1}{n}} \quad (27)$$

Therefore, from effective phonon theory,  $\kappa \propto T^{1/6}, T^{1/4}$  and  $T^{3/10}$  for  $H_3, H_4$  and  $H_5$  lattices, respectively. From Fig. 4, the numerical results of thermal conductivities for all these  $H_n$  lattices are plotted and compared with the theoretical predictions from effective phonon theory. Perfect agreements between theory and numerical results have been found for all the  $H_n$  lattices at temperature regions over three orders of magnitudes.

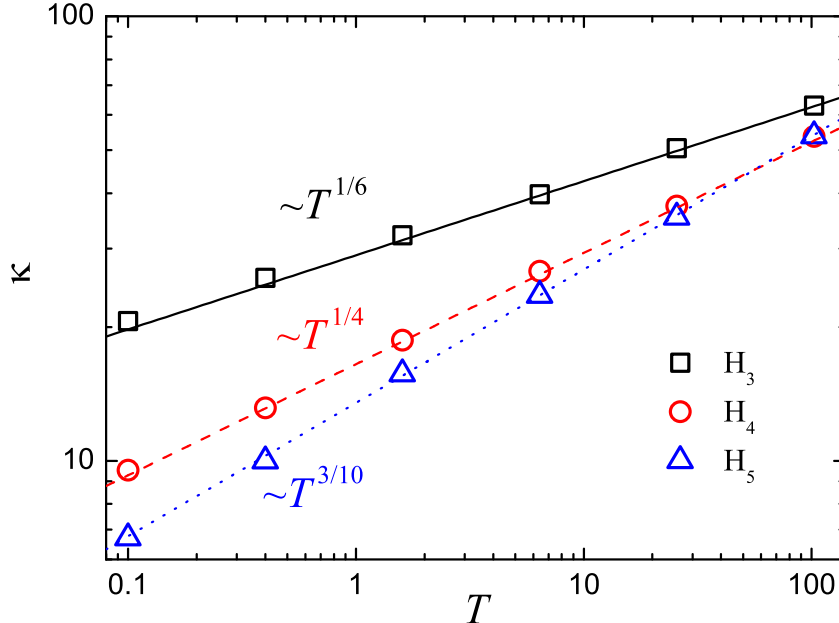


FIG. 4: (color online). Thermal conductivity  $\kappa$  as the function of temperature  $T$  for  $H_n$  lattices with  $n = 3, 4$  and  $5$ . The lattice sizes are all set as  $N = 500$ . The symbols are the numerical data from NEMD simulations.

#### IV. CONCLUSIONS

With this short review, we have briefly introduced the heat conduction of one-dimensional nonlinear lattices and the role of renormalized phonons and effective phonon theory. Although there are more than one type of excitation modes in the nonlinear lattices, detailed analysis on the temperature dependence of sound velocities reveals that the energy carriers responsible for the heat conduction should be the renormalized phonons.

Within the framework of renormalized phonons, a phenomenological theory, effective phonon theory, has been developed to explain the heat transport behaviors for general one-dimensional nonlinear lattices. For lattices without on-site potential, the effective phonon theory can *quantitatively* and *consistently* predict the scaling exponents of temperature-dependent thermal conductivities. In particular, for the FPU- $\beta$  lattices, the predictions from effective phonon theory are verified by NEMD simulations at both low and high temperature limits. For the special class of  $H_n$  lattices, the predictions from effective phonon theory are in perfect agreements with the obtained numerical results.

Finally, we need to point out that our work is closely related to the emerging field of phononics which focuses on the manipulation of heat flow and information processing with phonons[16]. For a two-segment thermal diode, the rectification of heat flow with the inversion of a temperature gradient is realized by the temperature-modulated overlap of the phonon spectra of two segments. It will be crucial to know the temperature dependence of the phonon spectrum in advance. Our analysis of the renormalized phonons due to nonlinearity or temperature enables us to provide the key information for the basic design of thermal diodes. On the other hand, our effective phonon theory dealing with the analysis of temperature-dependent thermal conductivities can be also very helpful for the design of novel phononics devices. The key element of phononics devices is the thermal transistor which relies on an effect of negative differential thermal resistance (NDTR). And this NDTR effect is a consequence of the system's very steep temperature-dependent thermal conductivity, e.g.  $\kappa \propto T^{\pm\eta}$  with  $\eta > 1$  at least. Therefore, we hope our work about renormalized phonons and effective phonon theory can provide solid theoretical support for the ongoing research filed of phononics[16].



## V. ACKNOWLEDGEMENTS

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